Homework 2 of Topology II
Due Date: Feb.7, 2018

1. Show that the boundary of any n-manifold is a $(n-1)$-manifold. (Construct the smooth structure for the boundary.)

2. Let Δ be the diagonal in $X \times X$. Show that the orthogonal complement to $T_{(x,x)}\Delta$ in $T_{(x,x)}(X \times X)$ is the collection of vectors $\{(v, -v) : v \in T_x X\}$.

3. (a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} exp(-\frac{1}{x^2}) & \text{if } x \geq 0 \\ 0 & \text{if } n \leq 0 \end{cases}$$

is smooth.

(b) There exists a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that h is identically zero on $(-\infty, 1]$; h is identically one on $[2, \infty)$; and $0 < h(x) < 1$ on $(1, 2)$.

(c) Show that there exists a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ with $supp(H)$ is contained in closed ball of 0 with radius 2 and identically 1 in the unit disk.

(d) Let M be a smooth manifold and $\{U_\alpha\}$ an open cover of M. Show that there exists a smooth partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$.

Definition: Partition of Unity on M subordinate to the open cover $\{U_\alpha\}$ is a collection of smooth functions $\{f_\alpha\}$ satisfying

(1) $0 \leq f_\alpha(x) \leq 1$ for all α and $x \in M$.

(2) $supp(f_\alpha) \subseteq U_\alpha$.

(3) For any $x \in M$, $f_\alpha(x) = 0$ except for finite many α.

(4) $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$.

4. Construct a smooth nonzero vector field on S^{2n-1} where n is any positive integer.

5. Use Brouwer's Fixed Point Theorem to show that every nonzero $n \times n$ matrix with non-negative real entries has a positive eigenvalue.

6. Show that the Brouwer Fixed Point Theorem fails for the open ball $B_n = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 < 1\}$. (Hint: Prove any open balls are diffeomorphic to the Euclidean space, and there are many maps on Euclidean space which have no fixed point.)